Math 141
Joseph C Foster
Summer 2017
Midterm 2

Name: Solutions.

June 1st, 2017
Time Limit: 55 minutes

This exam contains 10 pages (including this cover page) and 13 questions. The total number of points is 100. You have 55 minutes to complete the exam.

Read each question carefully. When specified, you must show **all** *necessary* work to receive full credit.

No calculator/phone/smartwatch allowed under any circumstances. Place these items in your bag, out of reach. Cheating of any kind will not be tolerated and will result in a grade of zero.

Question	Marks	Score	Question	Marks	Score
1	5		8	8	
2	5		9	18	
3	5		10	12	
4	5		11	12	
5	5		12	12	
6	5		13	12	
7	8		Total	100	

1. (5 marks) True or False: If $f(x)$ is continuous on an open interval (a, b), then f attains both an absolute maximum and an absolute minimum in (a, b).

   A. True

   B. False

2. (5 marks) Fill in the blank: Let $f(x)$ be continuous over the interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$ then there is at least one number $c \in (a, b)$ at which $f'(c) = 0$.

3. (5 marks) Fill in the blank: Let $f(x)$ be a differentiable function. Let $c \in \mathbb{R}$ be a point such that $f'(x) = 0$. Then;

   - If $f''(c) < 0$, $f(c)$ is a local maximum
   - If $f''(c) > 0$, $f(c)$ is a local minimum
   - If $f''(c) = 0$, $f(c)$ could be either.

For questions 4-6, choose the best answer. There is only one correct answer but you may choose up to *two*. If you choose two and one of the answers is correct, you will receive 2 marks.

4. (5 marks) What is the derivative of $f(x) = \sin^{-1}(x)$?

   A. $\dfrac{1}{\sqrt{1-x^2}}$

   B. $-\dfrac{1}{\sqrt{1-x^2}}$

   C. $\dfrac{1}{1+x^2}$

   D. $-\dfrac{1}{1+x^2}$

5. (5 marks) What is the derivative of $f(x) = \tan^{-1}(x)$?

   A. $\dfrac{1}{\sqrt{1-x^2}}$

   B. $-\dfrac{1}{\sqrt{1-x^2}}$

   C. $\dfrac{1}{1+x^2}$

   D. $-\dfrac{1}{1+x^2}$

6. (5 marks) Which of the following is the tangent line to the function $f(x) = x^2 + 1$ at the point $(2, 5)$?

   A. $y = 2x + 1$

   B. $y = 3x - 1$

   C. $y = 4x - 3$

   D. $y = 5x - 5$

In questions 7-17 you must show **all necessary work** to receive full credit. Unless specified, you do not need to simplify your final answer.

7. (8 marks) Prove that the function $f(x) = x^5 + x^3 - 1$ has *exactly* one real root.

$f(0) = 0^5 + 0^3 - 1 = -1$ $f(1) = 1^5 + 1^3 - 1 = 1$

Sign change, so since f is continuous, by IVT there is at least one root $x_0 \in (0,1)$.
If there were another root then Rolle's Theorem says there is a point $x_1 \neq x_0$ with $f'(x_1) = 0$ and $f'(x) < 0$ on $(x_1, x_1+\varepsilon)$ $f'(x) > 0$ on $(x_1 - \varepsilon, x_1)$ and (or vice versa). But since $f'(x) = 5x^4 + 3x^2 \geq 0$ this is not the case. Thus x_1 cannot exist and $f(x)$ has exactly one root.

8. (8 marks) Evaluate $\lim_{x\to\infty} (e^x + x)^{1/x}$

$$\lim_{x\to\infty} (e^x + x)^{1/x} = \lim_{x\to\infty} e^{\ln[(e^x+x)^{1/x}]}$$

$$= \lim_{x\to\infty} e^{\frac{\ln(e^x+x)}{x}} = e^{\lim_{x\to\infty} \frac{\ln(e^x+x)}{x}}$$

$$\underline{\underline{L'H}} \; e^{\lim_{x\to\infty} \frac{\frac{1}{e^x+x}(e^x+1)}{1}} = e^{\lim_{x\to\infty} \frac{e^x+1}{e^x+x}}$$

$$= e^{\lim_{x\to\infty} \frac{1+e^{-x}}{1+xe^{-x}}} = e^{\frac{1+\lim_{x\to\infty} e^{-x}}{1+\lim_{x\to\infty} xe^{-x}}}$$

$$= e^{\frac{1+0}{1+0}} = e^{\frac{1}{1}} = \boxed{e}$$

9. Let $f(x) = \dfrac{3(x^2+1)}{x^2-9}$. Then $f'(x) = -\dfrac{60x}{(x^2-9)^2}$ and $f''(x) = \dfrac{180(x^2+3)}{(x^2-9)^3}$. When asked for points, include *both* the x and y coordinate.

(a) i. (1 mark) Does the curve intercept the x-axis? If so, give the point(s) of interception. If not, say why.

No since $3(x^2+1) > 0$ for all x.

ii. (1 mark) Does the curve intercept the y-axis? If so, give the point(s) of interception. If not, say why.

$$f(0) = \frac{3(0+1)}{0-9} = \frac{3}{-9} = -\frac{1}{3} \qquad \boxed{(0, -1/3)}$$

(b) i. (1 mark) Does the curve have any vertical asymptotes? If so what are they?

$x^2 - 9 = 0 \implies x = \pm 3$

vertical asymptotes at $\boxed{x=3}$, $\boxed{x=-3}$

ii. (1 mark) Calculate $\lim_{x\to\infty} f(x)$. Does the curve have an asymptote at positive infinity? If so, what is it?

$\deg(3(x^2+1)) = \deg(x^2-9)$

$$\implies \lim_{x\to\infty} \frac{3(x^2+1)}{x^2-9} = 3.$$

Horizontal asymptote at infinity $\boxed{y=3}$

iii. (1 mark) Calculate $\lim_{x\to-\infty} f(x)$. Does the curve have an asymptote at negative infinity? If so, what is it?

$\deg(3(x^2+1)) = \deg(x^2-9)$

$$\implies \lim_{x\to\infty} \frac{3(x^2+1)}{x^2-9} = 3$$

Horizontal asymptote at negative infinity $\boxed{y=3}$

(c) i. (1 mark) For what values of x is $f(x)$ positive?

$3(x^2+1) > 0$ for all x

$x^2-9 > 0$ when $x < -3$, $x > 3$.

$f(x) > 0$ when $\boxed{x < -3}$ and $\boxed{x > 3}$

ii. (1 mark) For what values of x is $f(x)$ negative?

when $-3 < x < 3$

(d) i. (2 marks) What are the critical points of $f(x)$?

$f'(x) = 0 \implies 60x = 0 \implies x = 0$

$0^2 - 9 = -9 \neq 0$ Critical point at $\boxed{x=0}$

I did not deduct anything if these two weren't found.

$f'(x) = DNE \implies x = \pm 3$

Critical points at $\boxed{x = \pm 3}$

ii. (1 mark) For what values of x is $f(x)$ increasing?

$(x^2-9)^2 > 0$ for all x.

$-60x > 0$ when $x < 0$

$f(x) > 0$ when $\boxed{x < 0}$

iii. (1 mark) For what values of x is $f(x)$ decreasing?

when $x > 0$.

(e) (3 marks) Classify the critical point(s) you found above as either maximum or minimum.

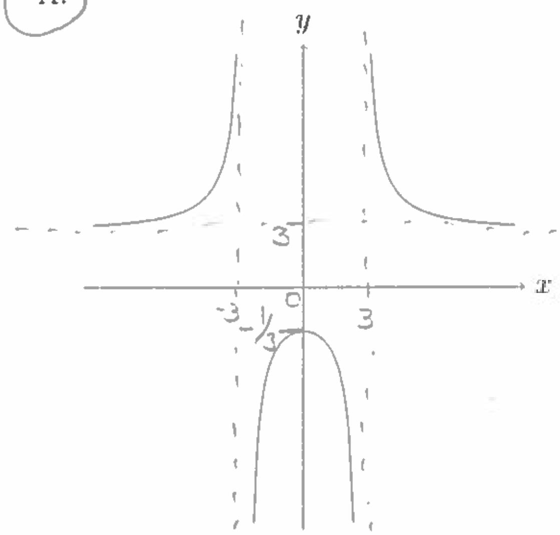
$$f''(0) = \frac{180(0+3)}{(0-9)^3} = \frac{180(3)}{-9^3} < 0$$

$x = 0$ is a (local) maximum

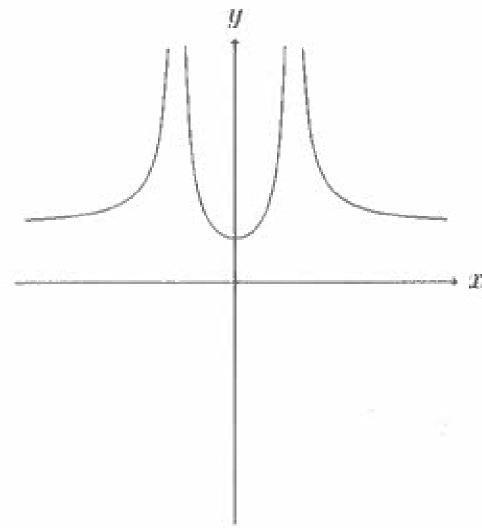
$f(x)$ is not defined at $x = \pm 3$ so they are neither.

(f) (4 marks) Which of the curves below represent $f(x)$. Choose only *one*. Label the points found above on your choice.

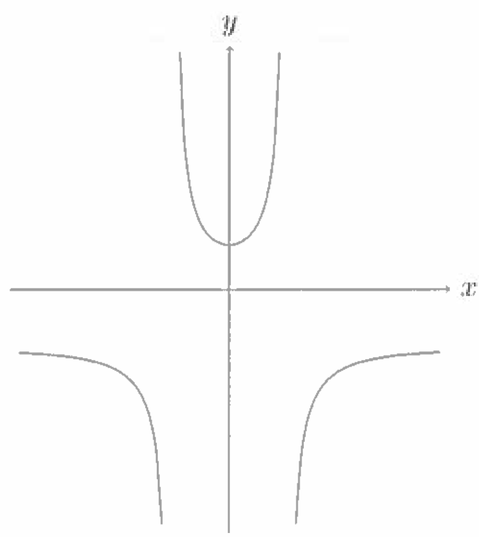
A.



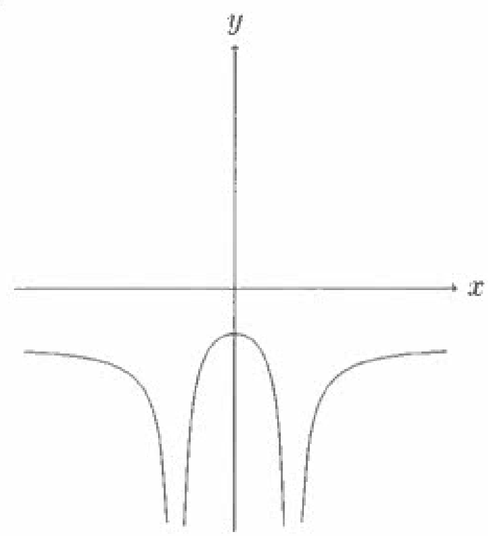
C.



B.

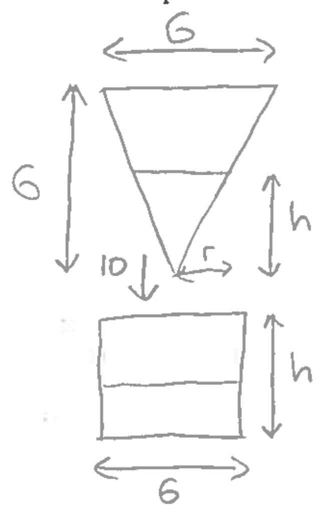


D.



Of questions 10 – 13, you must answer **three questions**. If you answer more than three then only **your best three scores will be counted**.

10. (12 marks) Coffee is draining from a conical filter, of height and diameter 6cm, into a cylindrical coffee pot, also of diameter 6cm, at a rate of 10cm^3/min. Calculate how fast the levels in both the pot and the cone are changing when the coffee in the cone is 5cm deep.

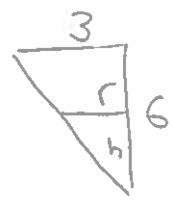


Pot

$$V = \pi r^2 h = 9\pi h$$

$$\frac{dV}{dt} = 9\pi \frac{dh}{dt}$$

$$\Longrightarrow \frac{dh}{dt} = \frac{1}{9\pi}\frac{dV}{dt} = \boxed{\frac{10}{9\pi}\ cm^3/min}$$



$$\frac{r}{h} = \frac{3}{6} \Longrightarrow r = \frac{h}{2}$$

Cone

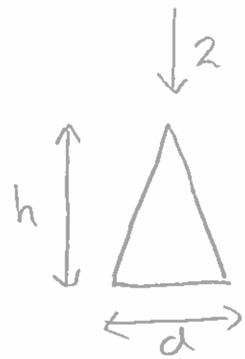
$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

$$\Longrightarrow \frac{dh}{dt} = \frac{4}{h^2\pi}\frac{dV}{dt} = \frac{-40}{h^2\pi}$$

$$h = 5 \Longrightarrow \frac{dh}{dt} = \frac{-40}{25\pi} = \boxed{\frac{-8}{5\pi}\ cm^3/min}$$

11. (12 marks) Assume that sand allowed to pour onto a level surface will form a pile in the shape of a cone, with height equal to the diameter of its base. If sand is poured at 2m^3/sec, how fast is the height of the pile changing when its base is 8m in diameter.



$$d = h \implies r = \frac{h}{2}.$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

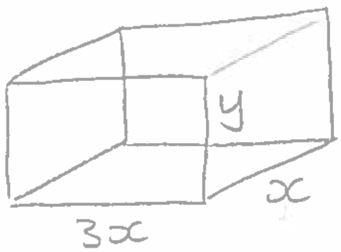
$$= \frac{\pi}{12} h^3$$

$$\implies \frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$

$$\implies \frac{dh}{dt} = \frac{4}{h^2\pi}\frac{dV}{dt} = \frac{8}{h^2\pi}$$

$$d = 8 \implies h = 8 \implies \frac{dh}{dt} = \frac{8}{8^2\pi} = \boxed{\frac{1}{8\pi} m^3/sec}$$

12. (12 marks) We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10 per square foot and the material used to build the sides cost \$6 per square foot. If the box must have a volume of 50ft^3, determine the dimensions that will minimise the cost to build the box.



$$C = 6(2xy + 2(3xy)) + 10(2)(3x^2)$$
$$= 48xy + 60x^2$$

$$V = 3x \times x \times y = 3x^2y = 50$$
$$\Longrightarrow y = \frac{50}{3x^2}$$

$$\Longrightarrow C = 48x\left(\frac{50}{3x^2}\right) + 60x^2$$
$$= \frac{16 \times 50}{x} + 60x^2$$

$$\Longrightarrow C' = -\frac{16 \times 50}{x^2} + 120x = 0$$

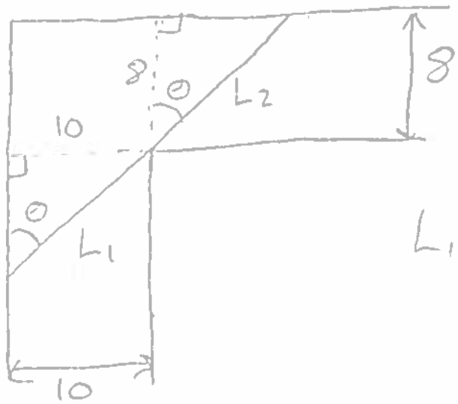
$$\Longrightarrow 120x = \frac{16 \times 50}{x^2}$$

$$\Longrightarrow x^3 = \frac{16 \times 50}{120} = \frac{16 \times 5}{12}$$
$$= \frac{4 \times 5}{3} = \frac{20}{3}$$

$$\Longrightarrow \boxed{x = \sqrt[3]{\frac{20}{3}}}$$

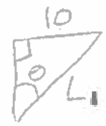
$$\Longrightarrow \boxed{y = \frac{50}{3\sqrt[3]{\frac{20}{3}}^{2}}} \left(= \frac{50}{\sqrt[3]{1200}} \right.$$
$$= \sqrt[3]{\frac{125000}{1200}}$$
$$\left. = 5\sqrt[3]{\frac{5}{6}} \right)$$

13. (12 marks) A piece of pipe is being carried down a hallway that is 10 feet wide. At the end of the hallway there is a right-angled turn and the hallway narrows to 8 feet wide. What is the longest pipe that can be carried (always keeping it horizontal) around the turn in the hallway?



$L = L_1 + L_2$

L_1:



$\sin(\theta) = \dfrac{opp}{adj} = \dfrac{10}{L_1}$

$\Longrightarrow L_1 = \dfrac{10}{\sin(\theta)} = 10\csc(\theta)$

L_2:

$\cos(\theta) = \dfrac{adj}{hyp} = \dfrac{8}{L_2}$

$\Longrightarrow L_2 = \dfrac{8}{\cos(\theta)} = 8\sec(\theta)$

$L = 10\csc(\theta) + 8\sec(\theta)$

$L' = -10\csc(\theta)\cot(\theta) + 8\sec(\theta)\tan(\theta) = 0$

$\Longrightarrow 10\csc(\theta)\cot(\theta) = 8\sec(\theta)\tan(\theta)$

$\Longrightarrow 10\csc(\theta) = 8\sec(\theta)\tan^2(\theta)$

$\Longrightarrow 10 = 8\tan^3(\theta)$

$\Longrightarrow \tan^3(\theta) = \dfrac{5}{4} \Longrightarrow \boxed{\theta = \tan^{-1}\left(\sqrt[3]{5/4}\right)}$

$\Longrightarrow L = 10\csc\left(\tan^{-1}\left(\sqrt[3]{5/4}\right)\right) + 8\sec\left(\tan^{-1}\left(\sqrt[3]{5/4}\right)\right)$